

The mod-2 cohomology of $32\Gamma_3f$

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We compute the group cohomology of $32\Gamma_3f$, a certain group of order 32. For this we construct explicit cocycle representatives of the cohomology generators. We thus lay to rest a discrepancy between several published computations of this cohomology ring.

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1 Introduction

We want to examine the group $32\Gamma_3f$ given by

$$\langle x, y \mid y^8 = 1, x^4 = y^4, {}^xy = y^3 \rangle.$$

This is a group of size 32 and the notation $32\Gamma_3f$ is taken from Hall–Senior [7]. Alternatively it is the group number 15 of size 32 in the numbering of the Small Groups library [2]. The above presentation is as in [8].

The mod-2-cohomology of $32\Gamma_3f$ has been calculated four times already, but with two incompatible results. Specifically Rusin [9] and Huebschmann[8] state that all cohomology classes in degree 3 are nilpotent, whereas Carlson [3] and Green–King [6, 5] find that there is an indecomposable element in degree 3 which is non-nilpotent. Here we present an independent computation which verifies the result of Carlson and Green–King; in particular we exhibit a standard 3-cocycle whose cohomology class is non-nilpotent.

The group admits the following presentation, which we are going to use throughout the article.

$$32\Gamma_3f = \left\langle f_1, f_2, f_3, f_4, f_5 \left| \begin{array}{l} f_1^2 = f_4, f_2^2 = f_3, f_3^2 = f_4^2 = f_5, f_5^2 = 1, \\ f_2^{f_1} = f_2f_3, f_3^{f_1} = f_3f_5, f_i^{f_j} = f_i \text{ for all other } j < i \end{array} \right. \right\rangle$$

$$\stackrel{\text{as set}}{=} \left\{ f_1^a f_2^b f_3^c f_4^d f_5^e \mid a, b, c, d, e \in \mathbb{F}_2 \right\}$$

This compares to the first presentation as follows: $f_1 \stackrel{\wedge}{=} x$, $f_2 \stackrel{\wedge}{=} y$, $f_3 \stackrel{\wedge}{=} y^2$, $f_4 \stackrel{\wedge}{=} x^2$ and $f_5 \stackrel{\wedge}{=} x^4 = y^4$. The operation is then given as

$$g_1 g_2 = (f_1^{a_1} f_2^{b_1} f_3^{c_1} f_4^{d_1} f_5^{e_1}) (f_1^{a_2} f_2^{b_2} f_3^{c_2} f_4^{d_2} f_5^{e_2}) = f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+\tilde{c}} f_4^{d_1+d_2+\tilde{d}} f_5^{e_1+e_2+\tilde{e}}$$

where

$$\tilde{c} = b_1 a_2 + b_1 b_2$$

$$\tilde{d} = a_1 a_2$$

$$\tilde{e} = d_1 d_2 + a_1 a_2 d_2 + a_1 d_1 a_2 + c_1 c_2 + b_1 b_2 c_2 + b_1 c_1 b_2 + b_1 a_2 c_2 + b_1 c_1 a_2 + c_1 a_2 + b_1 a_2 b_2$$

are the deviations from the elementary abelian case.

Along with [4, Conjecture 3] we propose the following conjecture.

Conjecture: The family of groups $\Phi_n := \langle x, y \mid y^{2^n} = 1, x^4 = y^{2^{n-1}}, xy = y^{2^{n-1}} \rangle \cong \langle x, y \mid y^{2^n} = 1, x^4 = y^{2^{n-1}}, xy = y^{2^{n-1}-1} \rangle$ for $n \geq 3$ is the only exception to Huebschmanns results and all groups of the family have the cohomology

$$H^*(\Phi_n) \cong \mathbb{F}_2[u_1, v_1, W_2, y_3, z_4] / (v^2 + uv = u^2 = uW = y^2 + W^3 = uy = 0).$$

Thus case (2) of Theorem E of [8] splits into two cases, the Φ_n with the corrected cohomology and the remainder with the already correct cohomology.

2 Breaking the problem apart

We describe $32\Gamma_3 f$ as a central extension

$$1 \rightarrow C_2 = \langle f_5 \rangle \hookrightarrow 32\Gamma_3 f \twoheadrightarrow 16\Gamma_2 c_2 \rightarrow 1 \quad (\text{I})$$

where the quotient is the group $16\Gamma_2 c_2$ of size 16 according to Hall–Senior or the group number 4 of size 16 according to the Small Groups Library. We describe $16\Gamma_2 c_2$ as

$$16\Gamma_2 c_2 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_5 = 1 \rangle$$

where the f_i otherwise behave as in the case of $32\Gamma_3 f$. Furthermore we describe $16\Gamma_2 c_2$ as a central extension

$$1 \rightarrow C_2 = \langle f_4 \rangle \hookrightarrow 16\Gamma_2 c_2 \twoheadrightarrow D_8 \rightarrow 1 \quad (\text{II})$$

where

$$D_8 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_4 = f_5 = 1 \rangle$$

is easily identifiable as a dihedral group of order 8.

3 Cohomology

We denote by $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$, the set maps $32\Gamma_3 f \rightarrow \mathbb{F}_2$ mapping $f_1^a f_2^b f_3^c f_4^d f_5^e$ to the respective exponent. We now utilize the bar construction to describe the cohomology of the group G ($32\Gamma_3 f$ or a quotient thereof), so that in degree n the cochains are given by the vector space of maps $G^n \rightarrow \mathbb{F}_2$ as given above, but with indices to indicate the relevant copy of

G . Thus for example $\bar{b}_1 \bar{a}_2 \bar{c}_2 : G^2 \rightarrow \mathbb{F}_2 : (g_1, g_2) = (f_1^{a_1} f_2^{b_1} f_3^{c_1} f_4^{d_1} f_5^{e_1}, f_1^{a_2} f_2^{b_2} f_3^{c_2} f_4^{d_2} f_5^{e_2}) \mapsto b_1 a_2 c_2$.

The cohomology of the D_8 is (see for example [1, Theorem 2.7], where the group generators are different so that our u is the sum of their degree one cohomology generators)

$$H^*(D_8) = \mathbb{F}_2[u_1, v_1, w_2]/(v^2 + uv = 0)$$

where the classes of degree one are given by $u = [\bar{a}]$, $v = [\bar{b}]$ with square brackets denoting the appropriate equivalence classes and the subscripts denoting the degree. Further we calculate for w in degree two (as $\langle f_2, f_3 \rangle$ is a C_4 we have $w \approx [\bar{c}^2]$).

$$\begin{aligned} \delta(\bar{c}_1 \bar{c}_2 + \bar{b}_1 \bar{a}_2 \bar{c}_2 + \bar{b}_1 \bar{c}_1 \bar{a}_2 + \bar{b}_1 \bar{b}_2 \bar{c}_2 + \bar{b}_1 \bar{c}_1 \bar{b}_2 + \bar{c}_1 \bar{a}_2 + \bar{b}_1 \bar{a}_2 \bar{b}_2)(g_1, g_2, g_3) &=: \delta(\Xi)(g_1, g_2, g_3) \\ &= \Xi(g_2, g_3) - \Xi(g_1 g_2, g_3) + \Xi(g_1, g_2 g_3) - \Xi(g_1, g_2) \\ &= (c_2 c_3 + b_2 a_3 c_3 + b_2 c_2 a_3 + b_2 b_3 c_3 + b_2 c_2 b_3 + c_2 a_3 + b_2 a_3 b_3) \\ &\quad - ((c_1 + c_2 + b_1 a_2 + b_1 b_2) c_3 + (b_1 + b_2) a_3 c_3 + (b_1 + b_2)(c_1 + c_2 + b_1 a_2 + b_1 b_2) a_3 \\ &\quad + (b_1 + b_2) b_3 c_3 + (b_1 + b_2)(c_1 + c_2 + b_1 a_2 + b_1 b_2) b_3 \\ &\quad + (c_1 + c_2 + b_1 a_2 + b_1 b_2) a_3 + (b_1 + b_2) a_3 b_3) \\ &\quad + (c_1(c_2 + c_3 + b_2 a_3 + b_2 b_3) + b_1(a_2 + a_3)(c_2 + c_3 + b_2 a_3 + b_2 b_3) + b_1 c_1(a_2 + a_3) \\ &\quad + b_1(b_2 + b_3)(c_2 + c_3 + b_2 a_3 + b_2 b_3) + b_1 c_1(b_2 + b_3) + c_1(a_2 + a_3) \\ &\quad + b_1(a_2 + a_3)(b_2 + b_3)) \\ &\quad - (c_1 c_2 + b_1 a_2 c_2 + b_1 c_1 a_2 + b_1 b_2 c_2 + b_1 c_1 b_2 + c_1 a_2 + b_1 a_2 b_2) \\ &= 0. \end{aligned}$$

Hence $w = [\bar{c}_1 \bar{c}_2 + \bar{b}_1 \bar{a}_2 \bar{c}_2 + \bar{b}_1 \bar{c}_1 \bar{a}_2 + \bar{b}_1 \bar{b}_2 \bar{c}_2 + \bar{b}_1 \bar{c}_1 \bar{b}_2 + \bar{c}_1 \bar{a}_2 + \bar{b}_1 \bar{a}_2 \bar{b}_2]$.

Furthermore the only non-trivial Steenrod operation is $Sq^1(w) = uw$ (see again [1, Theorem 2.7]).

Now we can determine the cocycle of the extension (II). To do this we select a splitting $\sigma : D_8 \rightarrow 16\Gamma_2 C_2$ on the set level and compute $q : D_8^2 \rightarrow \langle f_4 \rangle : (g_1, g_2) \mapsto \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$, which is a representative for the cocycle in $H^2(D_8, \langle f_4 \rangle \cong \mathbb{F}_2)$. We choose the canonical splitting mapping $f_1^a f_2^b f_3^c$ to $f_1^a f_2^b f_3^c f_4^0$. Thus we get

$$\begin{aligned} q(g_1, g_2) &= q(f_1^{a_1} f_2^{b_1} f_3^{c_1}, f_1^{a_2} f_2^{b_2} f_3^{c_2}) \\ &= \sigma(f_1^{a_1} f_2^{b_1} f_3^{c_1}) \sigma(f_1^{a_2} f_2^{b_2} f_3^{c_2}) \sigma(f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2})^{-1} \\ &= f_1^{a_1} f_2^{b_1} f_3^{c_1} f_1^{a_2} f_2^{b_2} f_3^{c_2} (f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2})^{-1} \\ &= f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2} f_4^{a_1 a_2} (f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2})^{-1} \\ &= f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2} (f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1 a_2+b_1 b_2})^{-1} f_4^{a_1 a_2} \\ &= f_4^{a_1 a_2} \end{aligned}$$

and the cocycle α is $[\bar{a}]^2 = u^2$.

Now we use the Leray-Serre-spectral-sequence to determine the cohomology. We have $E_2^{pq} = H^p(D_8) \otimes H^q(\langle f_4 \rangle) \implies H^*(16\Gamma_2 C_2)$, where $H^*(\langle f_4 \rangle) = H^*(C_2) = \mathbb{F}_2[t_1]$ (and

5	\vdots					
2	t^2	\dots				
1	t	ut, vt	\dots			
0	1	u, v	$u^2, uv = v^2, w$	u^3, v^3, uw, vw	$u^4, v^4, u^2w, w^2, v^2w$	\dots
	0	1	2	3	4	

$t = [\bar{d}]$). Now the cocycle $u^2 \in H^2(D_8)$ has to be killed and hence $d_2(t) = u^2$. Now by the algebra structure we can compute d_2 everywhere and get $E_3 \cong (H^*(D_8)/(u^2)) \otimes \mathbb{F}_2[t^2]$. Now we use the Steenrod operations to see by Kudo's transgression theorem

$$d_3(t^2) = d_3(Sq^1(t)) = Sq^1(d_2(t)) = Sq^1(u^2) = 0.$$

By the algebra structure all of d_3 is zero. Furthermore $d_n(t^2)$ for $n > 3$ vanishes trivially. Thus we have a collapse at the E_3 page and retrieve

$$H^*(16\Gamma_2 c_2) \cong \mathbb{F}_2[u_1, v_1, w_2, x_2]/(v^2 + uv = u^2 = 0).$$

Here u, v and w are simply inflated from $H^*(D_8)$.

Now x is roughly $[\bar{d}^2]$ and we verify

$$\begin{aligned}
& \delta(\bar{d}_1 \bar{d}_2 + \bar{a}_1 \bar{a}_2 \bar{d}_2 + \bar{a}_1 \bar{d}_1 \bar{a}_2)(g_1, g_2, g_3) \\
&= (d_2 d_3 + a_2 a_3 d_3 + a_2 d_2 a_3) \\
&\quad - ((d_1 + d_2 + a_1 a_2) d_3 + (a_1 + a_2) a_3 d_3 + (a_1 + a_2)(d_1 + d_2 + a_1 a_2) a_3) \\
&\quad + (d_1(d_2 + d_3 + a_2 a_3) + a_1(a_2 + a_3)(d_2 + d_3 + a_2 a_3) + a_1 d_1(a_2 + a_3)) \\
&\quad - (d_1 d_2 + a_1 a_2 d_2 + a_1 d_1 a_2) \\
&= 0.
\end{aligned}$$

Thus $x = [\bar{d}_1 \bar{d}_2 + \bar{a}_1 \bar{a}_2 \bar{d}_2 + \bar{a}_1 \bar{d}_1 \bar{a}_2]$. Furthermore the non-trivial Steenrod operations are $Sq^1(w) = uw$ (still valid) and $Sq^1(x) = 0$. For the latter we have to do a bit of work, the result lies in degree three and hence is a linear combination of uw, vw, ux and vx . Now we restrict to the two possible $C_4 \times C_2$ subgroups. For $\langle f_1, f_3, f_4 \rangle$ with cohomology $\mathbb{F}_2[p_1, q_1, r_2]/(q^2 = 0)$ we get the restrictions

$$u \mapsto q, \quad v \mapsto 0, \quad w \mapsto p^2 + pq, \quad x \mapsto r$$

and for $\langle f_2, f_3, f_4 \rangle$ with cohomology $\mathbb{F}_2[p'_1, q'_1, r'_2]/(q'^2 = 0)$ we get the restrictions

$$u \mapsto 0, \quad v \mapsto q', \quad w \mapsto r', \quad x \mapsto p'^2.$$

Now $Sq^1(p'^2) = 0 = Sq^1(r)$ (for the second one see e.g. [3, § 7.4]). Hence $Sq^1(x)$ cannot contain any of the four aforementioned constituents and thus must vanish.

3.1 Final computation for $32\Gamma_3 f$

We determine the extension cocycle of (I) as previously.

$$\begin{aligned} q(g_1, g_2) &= \sigma(f_1^{a_1} f_2^{b_1} f_3^{c_1} f_4^{d_1}) \sigma(f_1^{a_2} f_2^{b_2} f_3^{c_2} f_4^{d_2}) \sigma(f_1^{a_1+a_2} f_2^{b_1+b_2} f_3^{c_1+c_2+b_1a_2+b_1b_2} f_4^{d_1+d_2+a_1a_2})^{-1} \\ &= f_5^{d_1d_2+a_1a_2d_2+a_1d_1a_2+c_1c_2+b_1b_2c_2+b_1c_1b_2+b_1a_2c_2+b_1c_1a_2+c_1a_2+b_1a_2b_2} \end{aligned}$$

Thus we receive for the cocycle α now the value $w + x$.

5	\vdots					
4	t^4	\dots				
3	t^3	\dots				
2	t^2	ut^2, vt^2	$uv t^2 = v^2 t^2, wt^2, xt^2$	\dots		
1	t	ut, vt	$uv t = v^2 t, wt, xt$	\dots		
0	1	u, v	$uv = v^2, w, x$	uw, vw, ux, wx	$v^2 w, v^2 x, w^2, wx, x^2$	\dots
	0	1	2	3	4	5

Again we use the Leray-Serre-spectral-sequence to determine the cohomology. We have $E_2^{pq} = H^p(16\Gamma_2 c_2) \otimes H^q(\langle f_5 \rangle) \implies H^*(32\Gamma_3 f)$, where $H^*(\langle f_5 \rangle) = H^*(C_2) = \mathbb{F}_2[t_1]$ (and $t = [\bar{e}]$). Now the cocycle $w + x \in H^2(16\Gamma_2 c_2)$ has to be killed and hence $d_2(t) = w + x$. Now by the algebra structure we can compute d_2 everywhere and get $E_3 \cong (H^*(16\Gamma_2 c_2)/(w + x)) \otimes \mathbb{F}_2[t^2]$. Again we use Steenrod operations to determine d_3 as

$$d_3(t^2) = d_3(Sq^1(t)) = Sq^1(d_2(t)) = Sq^1(w + x) = uw.$$

We can thus determine d_3 everywhere. Note that $d_3(ut^2) = u^2 w = 0$ and $d_3(v^2 t^2) = v^2 uw = vu^2 w = 0$. This gives us $E_3 \cong (H^*(16\Gamma_2 c_2)/(w + x, uw)) \otimes \mathbb{F}_2[t^4]$. Again we use Steenrod operations for the next possible non-trivial differential

$$d_5(t^4) = d_5(Sq^2(t^2)) = Sq^2(d_3(t^2)) = Sq^2(uw) = u^3 w + uw^2 \stackrel{\wedge}{=} 0.$$

As before we see, that the spectral sequence collapses. We end up with the following E_∞ -page, where we omitted equal things due to $w = x$ and $uv = v^2$.

5	\vdots	\vdots	\vdots					
4	t^4	ut^4, vt^4	$wt^4, v^2 t^4, u^2 t^4 = 0$	\dots				
3								
2		ut^2	$v^2 t^2, u^2 t^2 = 0$	wut^2	$vwut^2$	\dots		
1								
0	1	u, v	v^2, w	vw	w^2	vw^2	w^3	\dots
	0	1	2	3	4	5	6	7

Now we have the generators u_1, v_1, W_2 (w equalling x), y_3 given by ut^2 and z_4 given by t^4 . We also get the relations for u, v and W and furthermore z has no relations. However

the relations involving y still need to be determined, since we have $y^2 \stackrel{\Delta}{=} u^2 t^4 = 0$ and $uy \stackrel{\Delta}{=} u^2 t^2 = 0$.

Now y corresponds roughly to $\bar{a}\bar{e}^2$. Indeed with $x_{12} := \bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2$ and $w_{12} := \bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2$ one can verify that the differential of

$$\begin{aligned} & \left(x_{12}w_{12} + (\bar{e}_1 + \bar{e}_2)(x_{12} + w_{12}) + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{c}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{d}_2 + \bar{c}_1\bar{a}_2\bar{c}_2 + \bar{e}_1\bar{a}_2 + \bar{e}_1\bar{e}_2 \right) \bar{a}_3 \\ & + x_{12}\bar{d}_3 \end{aligned}$$

vanishes. Thus we can use this term as representative for y .

To determine the relations of y we use the subgroup $K := \langle f_2, f_3, f_4, f_5 \rangle \cong C_8 \times C_2$, where the generators are f_2 and f_3f_4 . It has cohomology $H^*(K) \cong \mathbb{F}_2[\xi_1, \varphi_1, \chi_2]/(\varphi^2)$, where ξ belongs to the C_2 .

We first compute the restrictions on the chain level. The inclusion is $K \hookrightarrow G : f_2^i f_3^j f_5^k (f_3 f_4)^l \mapsto f_2^i f_3^{j+l} f_4^l f_5^k$ giving restrictions

$$\bar{a} \mapsto 0, \quad \bar{b} \mapsto \bar{i}, \quad \bar{c} \mapsto \bar{j} + \bar{l}, \quad \bar{d} \mapsto \bar{l}, \quad \bar{e} \mapsto \bar{k}.$$

The cohomology classes of further interest are $\xi = [\bar{l}]$ and $\varphi = [\bar{i}]$. With them we get restrictions (suppressing most terms which map to zero because they contain \bar{a})

$$\begin{aligned} u = [\bar{a}] &\mapsto 0, \\ v = [\bar{b}] &\mapsto [\bar{i}] = \varphi, \\ W = [\bar{d}^2 + \bar{a}(\dots)] &\mapsto [\bar{l}^2] = \xi^2, \\ y = [\bar{d}^3 + \bar{a}(\dots)] &\mapsto [\bar{l}^3] = \xi^3. \end{aligned}$$

Now we look at the spectral sequence and see that during ungrading we get the following uncertainties: y^2 is a linear combination of W^3 and vWy whereas uy is a scalar multiple of W^2 .

We restrict to K and immediately retrieve $y^2 = W^3$ and $uy = 0$. Thus we get the following.

Theorem: The mod-2-cohomology ring of $32\Gamma_3 f$ is

$$H^*(32\Gamma_3 f, \mathbb{F}_2) \cong \mathbb{F}_2[u_1, v_1, W_2, y_3, z_4]/(v^2 + uv = u^2 = uW = y^2 + W^3 = uy = 0).$$

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